

# On the Structure Constants of Volume Preserving Diffeomorphism Algebra

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## Abstract

Regularizing volume preserving diffeomorphism (VPD) is equivalent to a long standing problem, namely regularizing Nambu-Poisson bracket. In this paper, as a first step to regularizing VPD, we find general complete independent basis of VPD algebra. Especially, we find complete independent basis that give simple structure constants, where three area preserving diffeomorphism (APD) algebras are manifest. This implies that an algebra that regularizes VPD algebra should include three  $u(N)$  Lie algebras.

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# 1 Introduction

Area preserving diffeomorphism (APD) algebra is regularized by  $u(N)$  Lie algebra. Actually, a large  $N$  limit of structure constants of  $u(N)$  Lie algebra in the 't Hooft basis reduces to those of APD algebra defined on  $T^2$  [1–4]. Because APD algebra is generated by Poisson bracket, it is regularized by Lie bracket of  $u(N)$  Lie algebra. This structure induces that the Heisenberg picture of quantum mechanics reduces to the canonical formalism of classical mechanics in the classical limit. Another application is that one can show that BFSS matrix theory and IIB matrix model contain the lightcone supermembrane and the type IIB superstring, respectively by using this regularization [5–7].

On the other hand, regularizing Nambu-Poisson bracket is a long standing problem<sup>1</sup> [14–37]. As in the case of APD, Nambu-Poisson bracket generates volume preserving diffeomorphism (VPD) algebra. In this paper, as a first step to regularizing Nambu-Poisson bracket, we search for several independent basis of VPD algebra and obtain simple structure constants.

## 2 General Complete Independent Basis of VPD Algebra

VPD is a diffeomorphism  $x^i \rightarrow y^i(x)$  ( $i = 1, 2, 3$ ) that satisfies  $\det \partial_i y^j(x) = 1$ . Then, the infinitesimal transformation  $y^i(x) = x^i + \delta x^i(x)$  satisfies

$$\partial_i \delta x^i(x) = 0. \quad (2.1)$$

$\delta x^i(x) = \epsilon^{ijk} \partial_j f(x) \partial_k g(x)$  satisfy this equation. Transformations of a scalar field generated by these solutions are given by

$$\begin{aligned} \delta Z(x) &\equiv \delta x^i(x) \partial_i Z(x) \\ &= \epsilon^{ijk} \partial_i f(x) \partial_j g(x) \partial_k Z(x) \\ &= \{f(x), g(x), Z(x)\}. \end{aligned} \quad (2.2)$$

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<sup>1</sup>For example, if the problem is solved, one should be possible to show that a three algebra model of M-theory [8–13] contains the semi-lightcone supermembrane.

This implies that Nambu-Poisson bracket generates VPD. The transformations

$$\delta = \delta x^i(x) \partial_i = \epsilon^{ijk} \partial_i f(x) \partial_j g(x) \partial_k \quad (2.3)$$

form VPD algebra.

APD is a two-dimensional analogue of VPD. Infinitesimal transformations

$$\delta = \delta X^I(Y) \partial_I = \epsilon^{IJ} \partial_I F(Y) \partial_J \quad (2.4)$$

on  $T^2$  ( $I, J = 1, 2$ ), where

$$\partial_I \delta X^I(X) = 0, \quad (2.5)$$

are spanned by generators

$$\delta(A) = ie^{iAY} \epsilon^{IJ} A_I \partial_J, \quad (2.6)$$

which are obtained by substituting  $F(Y) = e^{iAY}$  to (2.4).

On the other hand, complete independent basis of VPD cannot be obtained by substituting  $f(x) = e^{iax}$  and  $g(x) = e^{ibx}$  to (2.3) because  $\delta x^i(x)$  is a local vector in three dimensions. We need to solve (2.1). In the case of APD, (2.6) are complete independent solutions of (2.5). On  $T^3$ , we make a Fourier transformation,  $\delta x^i(x) = \sum_a v^i(a) e^{iax}$ . (2.1) implies

$$a_i v^i(a) = 0. \quad (2.7)$$

An independent solution of (2.7) is given by

$$\begin{aligned} \bar{v}_1 &= (-a_2, a_1, 0) \\ \bar{v}_2 &= a \times \bar{v}_1 = (-a_1 a_3, -a_2 a_3, a_1^2 + a_2^2), \end{aligned} \quad (2.8)$$

for  $a = (a_1, a_2, a_3)$  (except  $a_1 = a_2 = 0$ ), and

$$\begin{aligned} \bar{v}'_1 &= (1, 0, 0) \\ \bar{v}'_2 &= (0, 1, 0), \end{aligned} \quad (2.9)$$

for  $a = (0, 0, a_3)$ .

The corresponding VPD generators are given by

$$\begin{aligned} S_1(a) &= e^{iax} \bar{v}_1^i \partial_i = e^{iax} (-a_2 \partial_1 + a_1 \partial_2) \\ S_2(a) &= e^{iax} \bar{v}_2^i \partial_i = e^{iax} (-a_1 a_3 \partial_1 - a_2 a_3 \partial_2 + (a_1^2 + a_2^2) \partial_3) \end{aligned} \quad (2.10)$$

$$\begin{aligned} S'_1(0, 0, a_3) &= e^{ia_3 x^3} \bar{v}'_1^i \partial_i = e^{ia_3 x^3} \partial_1 \\ S'_2(0, 0, a_3) &= e^{ia_3 x^3} \bar{v}'_2^i \partial_i = e^{ia_3 x^3} \partial_2, \end{aligned} \quad (2.11)$$

which form VPD algebra

$$\begin{aligned}
[S_1(a), S_1(b)] &= i(a_1b_2 - a_2b_1)S_1(a+b) \\
[S_2(a), S_2(b)] &= i\alpha S_1(a+b) + i\beta S_2(a+b) \\
[S_1(a), S_2(b)] &= i\gamma S_1(a+b) + i\delta S_2(a+b) \\
[S_1(a), S'_1(0, 0, b_3)] &= -ia_1S_1(a+b) \\
[S_2(a), S'_1(0, 0, b_3)] &= -ia_2b_3S_1(a+b) - ia_1S_2(a+b) \\
[S_1(a), S'_2(0, 0, b_3)] &= -ia_2S_1(a+b) \\
[S_2(a), S'_2(0, 0, b_3)] &= ia_1b_3S_1(a+b) - ia_2S_2(a+b) \\
[S'_1(0, 0, a_3), S'_1(0, 0, b_3)] &= [S'_2(0, 0, a_3), S'_2(0, 0, b_3)] = [S'_1(0, 0, a_3), S'_2(0, 0, b_3)] = 0,
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
\alpha &= \frac{1}{(a_1 + b_1)^2 + (a_2 + b_2)^2} (a_2b_1 - a_1b_2)((a_1^2 + a_2^2)b_3^2 + (b_1^2 + b_2^2)a_3^2 - 2a_3b_3(a_1b_1 + a_2b_2)) \\
\beta &= \frac{1}{(a_1 + b_1)^2 + (a_2 + b_2)^2} ((a_1^2 + a_2^2)b_3((a_1 + b_1)b_1 + (a_2 + b_2)b_2) \\
&\quad - (b_1^2 + b_2^2)a_3((a_1 + b_1)a_1 + (a_2 + b_2)a_2)) \\
\gamma &= \frac{1}{a_1 + b_1} \left( \frac{1}{(a_1 + b_1)^2 + (a_2 + b_2)^2} (b_1^2 + b_2^2)(a_1b_2 - a_2b_1)(a_2 + b_2)(a_3 + b_3) \right. \\
&\quad \left. - b_2b_3(a_1b_2 - a_2b_1) - a_1(-b_3(a_1b_1 + a_2b_2) + a_3(b_1^2 + b_2^2)) \right) \\
\delta &= \frac{1}{(a_1 + b_1)^2 + (a_2 + b_2)^2} (b_1^2 + b_2^2)(a_1b_2 - a_2b_1).
\end{aligned} \tag{2.13}$$

This algebra has a complicated form because the basis (2.10) and (2.11) are complicated.

General independent solutions of (2.7) are given by

$$\begin{aligned}
v_1^i &= \epsilon^{ijk} a_j l_k(a) \\
v_2^i &= \epsilon^{ijk} a_j m_k(a),
\end{aligned} \tag{2.14}$$

where  $a$ ,  $l(a)$ ,  $m(a)$  are all independent for all  $a$ . The corresponding generators are given by

$$\begin{aligned}
T_1(a) &:= e^{ia \cdot x} \det(la\partial) \\
T_2(a) &:= e^{ia \cdot x} \det(ma\partial),
\end{aligned} \tag{2.15}$$

where  $\det(abc) := \epsilon^{ijk} a_i b_j c_k$ .

If we choose  $l = (0, 0, 1)$  and  $m = (-a_2, a_1, 0)$  for  $a = (a_1, a_2, a_3)$  (except  $a_1 = a_2 = 0$ ), (2.14) and (2.15) represent (2.8) and (2.10), respectively. If we choose  $l = (0, -\frac{1}{a_3}, 0)$  and  $m = (\frac{1}{a_3}, 0, 0)$  for  $a = (0, 0, a_3)$ , (2.14) and (2.15) represent (2.9) and (2.11), respectively.

### 3 Simple Structure Constants of VPD Algebra

In this section, we search for complete independent basis that give more simple structure constants. Although (2.15) for constant  $l$  and  $m$  are not independent in a part of the region of  $a$  where  $a$  is on a plane spanned by  $l$  and  $m$ , we can calculate commutation relations among (2.15) for constant  $l$  and  $m$ , and then obtain simple relations

$$[T_1(a), T_1(b)] = i \det(lab) T_1(a+b) \quad (3.1)$$

$$[T_2(a), T_2(b)] = i \det(mab) T_2(a+b) \quad (3.2)$$

$$[T_1(a), T_2(b)] = i \frac{1}{\det(lm(a+b))} (\det(mab) \det(lma) T_1(a+b) + \det(lab) \det(lmb) T_2(a+b)). \quad (3.3)$$

For example, if we choose  $l = (0, 0, 1)$  and  $m = (1, 0, 0)$ , we obtain  $v_1 = (-a_2, a_1, 0)$ ,  $v_2 = (0, -a_3, a_2)$ , and the corresponding generators

$$\begin{aligned} U_1(a) &= e^{iax} (-a_2 \partial_1 + a_1 \partial_2) \\ U_2(a) &= e^{iax} (-a_3 \partial_2 + a_2 \partial_3). \end{aligned} \quad (3.4)$$

In this case, for  $a_2 = 0$ ,  $v_1 = (0, a_1, 0)$  and  $v_2 = (0, -a_3, 0)$  are dependent, and thus  $U_1(a)$  and  $U_2(a)$  are.

Then, we choose a step function,  $m = (1, 0, 0)$  for  $a_2 \neq 0$  and  $m = (0, 1, 0)$  for  $a_2 = 0$ . When  $m = (0, 1, 0)$  we have  $v_3 = (a_3, 0, -a_1)$  and

$$U_3(a) = e^{iax} (a_3 \partial_1 - a_1 \partial_3), \quad (3.5)$$

which is independent of  $U_1(a)$  and  $U_2(a)$  for  $a_2 = 0$ . After considering  $v_1 = 0$  when  $a_1 = a_2 = 0$ , we have a complete set of independent generators

$$\begin{aligned} &U_1(a) \text{ (except } a_1 = a_2 = 0) \\ &U_2(a) \text{ (except } a_2 = 0) \\ &U_2(0, 0, a_3) \\ &U_3(a_1, 0, a_3). \end{aligned} \quad (3.6)$$

In fact, for each  $a$  there are independent two generators;

$$\begin{aligned}
&U_1(a) \text{ and } U_2(a) \text{ for } a_2 \neq 0 \\
&U_1(a) \text{ and } U_3(a) \text{ for } a_2 = 0 \text{ and } a_1 \neq 0 \\
&U_2(a) \text{ and } U_3(a) \text{ for } a_2 = a_1 = 0.
\end{aligned} \tag{3.7}$$

Then, we obtain simple structure constants of VPD algebra,

$$[U_1(a), U_1(b)] = i(a_1b_2 - a_2b_1)U_1(a+b) \tag{3.8}$$

$$[U_2(a), U_2(b)] = i(a_2b_3 - a_3b_2)U_2(a+b) \tag{3.9}$$

$$[U_3(a_1, 0, a_3), U_3(b_1, 0, b_3)] = i(a_3b_1 - a_1b_3)U_3(a_1 + b_1, 0, a_3 + b_3) \tag{3.10}$$

$$[U_1(a), U_2(b)] = i \frac{1}{a_2 + b_2} (a_2(a_2b_3 - a_3b_2)U_1(a+b) + b_2(a_1b_2 - a_2b_1)U_2(a+b)) \tag{3.11}$$

$$[U_1(a), U_3(b_1, 0, b_3)] = i((-a_1b_3 + a_3b_1 + b_1b_3)U_1(a+b) + b_1^2U_2(a+b)) \tag{3.12}$$

$$[U_1(a), U_3(b_1, 0, b_3)] = i(-b_3^2U_1(a+b) + (a_3b_1 - b_3a_1 - b_1b_3)U_2(a+b)). \tag{3.13}$$

From (3.8), (3.9) and (3.10), one can see three APD algebras corresponding to  $(x^1, x^2)$ ,  $(x^2, x^3)$  and  $(x^3, x^1)$  planes.

## 4 Conclusion and Discussion

In this paper, we found general complete independent basis of VPD algebra. Especially, we found complete independent basis that give simple structure constants where three APD algebras are manifest. This implies that an algebra that regularizes VPD algebra should include three  $u(N)$  Lie algebras.

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